# Intro Video: Section 3.4
# the Chain Rule

Math F251X: Calculus 1

Ways to combine functions $f$ & $g$:

| $f+g$ | $fg$ | $\frac{f}{g}$ |
|---|---|---|
| $f(x)+g(x)$ | $f(x)\cdot g(x)$ | $\frac{f(x)}{g(x)}$ |

We know how to differentiate these!

| $f \circ g$ |
|---|
| $f(g(x))$ |

THE CHAIN RULE lets us differentiate this!

$\boxed{\text{Chain Rule:}}$

Version 1: $\frac{d}{dx}(f(g(x)) = f'(g(x))\, g'(x)$

The derivative of the outside, with respect to the inside, times the derivative of the inside.

Version 2: Think of $y = g(u)$, and $u = f(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

**Example:** $H(x) = \left(3x^2 + \sin(x)\right)^5$

$f(\square) = \square^5$

$g(x) = 3x^2 + \sin(x)$

$$H'(x) = f'\left(\square\right) \frac{d}{dx}\left(\square\right)$$

$$= 5\left(\square\right)^4 \left(6x + \cos(x)\right)$$

$$= 5\left(3x^2 + \sin(x)\right)\left(6x + \cos(x)\right)$$

$$\frac{dH}{dx} = \frac{dH}{du} \cdot \frac{du}{dx} = 5u^4 \cdot \frac{du}{dx} = \left(5u^4\right)\left(6x + \cos(x)\right)$$

$u = g(x)$

$$= 5\left(3x^2 + \sin(x)\right)\left(6x + \cos(x)\right)$$

Example: $y = \tan(x) + e^{\tan(x)}$ ...

$e^{\tan(x)} = \exp(\tan(x))$

$$y' = (\sec(x))^2 + e^{\tan(x)} \cdot \frac{d}{dx}(\tan(x))$$

$$= (\sec(x))^2 + e^{\tan(x)}(\sec(x))^2$$

Useful fact: $\frac{d}{dx}\left(e^{f(x)}\right)$ always requires the chain rule!

Example: $\frac{d}{dx}\left(e^{x^2+4x}\right) = e^{x^2+4x}\frac{d}{dx}(x^2+4x) = e^{x^2+4x}(2x+4)$

Example: $\frac{d}{dx}\left(e^{2x}\right) = e^{2x}(2)$

Example: $\frac{d}{d\theta}\left(\sin(4\theta)\right) = \cos(4\theta)(4)$

<u>Example</u>: $f(x) = \sqrt{5x} = (5x)^{1/2}$

① Use the chain rule:

$$f'(x) = \frac{1}{2}(5x)^{-1/2}(5) = \frac{5}{2\sqrt{5x}} = \frac{(\sqrt{5})^2}{2\sqrt{5}\sqrt{x}} = \frac{\sqrt{5}}{2\sqrt{x}}$$

② Do algebra:

$$f(x) = 5^{1/2}\, x^{1/2}$$

$$f'(x) = 5^{1/2}\, \frac{1}{2}\, x^{-1/2} = \frac{\sqrt{5}}{2\sqrt{x}}$$

# More examples

① $h(x) = \sec(\boxed{e^x + x^2})$

$u = e^x + x^2$

So $\dfrac{dh}{dx} = \dfrac{dh}{du} \cdot \dfrac{du}{dx}$

$$\frac{dh}{dx} = \sec(u)\tan(u) \cdot \boxed{\frac{du}{dx}}$$
$$= \sec(e^x + x^2)\tan(e^x + x^2)\,(e^x + 2x)$$

② $j(\theta) = \dfrac{\theta^3 - \cos\theta}{\tan(5\theta)}$

So $$j'(\theta) = \frac{(\tan(5\theta))\frac{d}{d\theta}(\theta^3 - \cos\theta) - (\theta^3 - \cos\theta)\frac{d}{d\theta}(\tan(5\theta))}{(\tan(5\theta))^2}$$

chain rule!

$$= \frac{\tan(5\theta)(3\theta^2 + \sin\theta) - (\theta^3 - \cos\theta)((\sec(5\theta))^2(5))}{(\tan(5\theta))^2}$$

**Example:** $g(t) = e^{\cos(7t-5)}$

**Example:** Let $L(t) = 12 + 2.8\sin\left(\frac{2\pi}{365}(t-80)\right)$

be # hours of daylight for an east coast city, where $t$ is # days since January 1.

→ What is the rate of change of # hours of daylight on March 21 & September 21?

$t = 79$ $\qquad$ $t = 263$

$$L'(t) = 2.8\cos\left(\frac{2\pi}{365}(t-80)\right)\frac{d}{dt}\left(\frac{2\pi}{365}(t-80)\right)$$

$$= 2.8\cos\left(\frac{2\pi}{365}(t-80)\right)\left(\frac{2\pi}{365}\right)$$

$L'(79) \approx 0.048$ and $L'(263) \approx -0.048$

Increasing daylight! $\qquad$ decreasing daylight

$$\left(\frac{0.048 \text{ hours}}{\text{day}}\right)\left(\frac{60 \text{ min}}{\text{hour}}\right) = 2.9 \text{ minutes/day}$$